NONCONTRACTIBLE HETEROGENEITY IN DIRECTED SEARCH

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Abstract. This paper provides a directed search model designed to explain the residual part of wage variation left over after the impact of education and other observable worker characteristics has been removed. Workers have private information about their characteristics at the time they apply for jobs. Firms value these characteristics differently and can observe them once workers apply. They hire the worker they most prefer. However, the characteristics aren't contractible, so firms can't condition their wages on them. The paper shows how to extend arguments from directed search to handle this, allowing for arbitrary distributions of worker and firm types. The model is used to provide a functional relationship that ties together the wage distribution and the wage-duration function. This relationship provides a testable implication of the model. This relationship suggests a common property of wage distributions that guarantees that workers who leave unemployment at the highest wages also have the shortest unemployment duration. This is in strict contrast to the usual (and somewhat implausible) directed search story in which high wages are always accompanied by higher probability of unemployment.

1. Introduction

This paper provides a directed search model in which worker and firm characteristics differ, but where firms cannot cannot condition the wages they pay on worker characteristics as they can in papers like in (Shi 2002) or (Shimer 2005). Examples of such characteristics might be things like reference letters that convey a lot of information about an applicants skill as long as they are not contractible. Another example might be connections and friendships that workers have with managers, or just with other workers in the industry. These connections typically cannot be verified in any way that would be satisfactory in a formal contract. Alternatively, firms may care a lot about worker

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characteristics on which they aren't allowed to condition wages - for example whether or not the worker has a criminal record, or a history of union activism.

Workers know their own characteristics at the time they apply for jobs, but they don't know the characteristics of other workers who might apply. Firms value these characteristics differently and can observe these characteristics once workers apply. Once they have collected a bunch of applications, they hire the worker they most prefer.

The paper shows how to extend directed search to handle this, allowing for arbitrary distributions of worker and firm types. More broadly, this approach provides a method to understand the variation in wages that cannot be attributed to observable characteristics. The basic logic of the model ties together the wage distribution and the unemployment duration function, i.e., the relationship between the wage at which a worker leaves unemployment, and his duration. This relationship provides a potentially testable implication of the model.

The relationship between the wage duration function and the wage distribution developed in this paper makes it possible to examine one of the key predictions of directed search in models where wages can't be conditioned on worker type - workers who submit applications to high wage firms should expect to be hired by those firms with low probability.[1] There isn't a lot of evidence about this central prediction, however, what there is doesn't seem to support it. Addison, Centeno, and Portugal (2004), for example, provide some evidence to suggest that the wages at which workers leave unemployment and the duration of their unemployment spell are negatively correlated. The evidence is not strong, but it certainly provides no support at all for the classical prediction of directed search.[2]

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[1]For example (Peters 2000), (Lang, Manove, and Dickens 1999), (Eeckhout and Kircher 2008),(Acemoglu and Shimer 2000) or (Shi 2009) all generate equilibrium wage dispersion for which wage and employment probability are related in this way. Of course, when wages can be made conditional on type, as in (Shi 2002) or (Shimer 2005), workers who receive higher wages are being compensated for having a more valuable type, not for bearing risk, so no such relationship would be expected in these models.

[2]Many models of directed search assume that workers and firms are identical, so this assertion isn't so much a prediction as it is a statement about what happens out of equilibrium. However, there are many models that support distributions of wages in equilibrium. For example, Shi (2002), Shimer (2005) and (Lang, Manove, and Dickens 1999) all allow for a finite number of different types of workers. In (Peters 2000) workers are identical, but there is a continuum of different types of firms. In Delacroix and Shi (2006) and (Shi 2009), workers equilibrium wages (and search strategies) depend on their employment history. Albrecht, Gautier, and Vroman

The argument below illustrates that the relationship between wage and unemployment duration is driven by two considerations. The first is completely intuitive - higher wage firms will tend to hire workers whose (externally unobservable) quality is higher, and these workers will tend to be more likely to find jobs no matter where they apply. This creates a positive relationship between quality, employment probability and wage. This is confounded by the fact that higher quality workers will tend to use different application strategies than low quality workers. In particular, they will tend to apply at high wage jobs along with a lot of other high quality workers. This effect leads in the opposite direction, lowering the probability with which high quality workers will be hired. This is where the directed search model plays a role since it ties down the application strategy for workers of different qualities. The characterization of the equilibrium application strategy provides a testable connection between wage offer distribution and the duration function. One implication of this relationship is that it provides a relatively simple (and apparently normally satisfied) restriction on the wage distribution that ensures that workers who leave unemployment at high wages tend to have shorter unemployment spells.

The paper begins with an analysis of the case where there is a continuum of workers and firms. The actions of individual firms support a wage offer distribution. Workers' decisions support a joint distribution of applications across wages and types. The payoff functions resemble those in a standard directed search model. Yet there is an important difference. Instead of being concerned with the expected number of competitors he will face when he applies at a given wage, a worker is instead concerned with the expected number of competitors with higher types.

For this reason, firms who set high wages don't necessarily get more applicants than low wage firms. However, the average quality of the applicant they receive is higher and this compensates them for the higher wages they commit themselves to pay. If firms differ in the way they value worker quality on the margin, then firms who like worker quality will set higher wages. This will induce an imperfect matching of high quality workers with firms who value that quality.

However, we show that this matching will be imperfect. The reason is that equilibrium application strategies will have workers acting as if

(2006) or Galenianos and Kircher (2005) support wage distributions by allowing multiple applications. Finally, other than the model described in the paper, the only one I am familiar with that allows a continuum of both firm and worker types is (Eeckhout and Kircher 2008). In all these models, the standard tradeoff between high wages and low employment probabilities occurs on the equilibrium path.

they were following a 'reservation wage rule' in which they apply with equal probability to all firms who set wages above their reservation wage. For this reason mismatches will continue, with high value employees remaining unemployed simply because they tried to compete with other high quality employees and high producitivity firms filling vacancies with low quality workers simply because no high quality workers apply.

We provide a pair of functional equations that can be used to characterize the equilibrium wage distribution and the equilibrium application strategy of workers. We use these to illustrate the equilibrium with a number of examples and to provide a condition that can be used to test the model. We also revisit the relationship mentioned above between unemployment duration and exit wage. From the arguments above, it should be apparent that exit wages induce a selection bias - workers who leave unemployment at high wages tend to be higher type workers who don't compete with the lower quality applicants who also apply to high wage firms. As a consequence, there is no reason for them to experience long unemployment spells. We use the functional equations to compute the relationship between exit wage and average duration. In general this relationship isn't systematic. However, we provide a restriction on the wage distribution that ensures that workers who leave unemployment at higher wages actually have higher employment probabilities.

In the final part of the paper, we focus on a micro foundation for the model. This argument justifies the particular payoffs used in the main part of the paper. It also illustrates the result that workers should follow a reservation wage strategy in choosing where to apply. We consider a finite array of firms and wages and imagine a finite number of workers with privately knows types who apply to these firms. We characterize the unique symmetric Bayesian equilibrium application strategy for workers. As mentioned, it involves a reservation wage, then a set of application probabilities with which workers apply to higher wage firms. As might be expected, the higher the wage (or the farther away it is from the worker's reservation wage) the lower the probability that the worker applies. At that point, we explain why it is that low type workers *have to* apply to high wage firms with some probability in order to support the equilibrium. This, of course, rules out assortative matching. One useful consequence of this argument is to distinguish models like this one, where firms care about the type of the worker they hire, from models like (Eeckhout and Kircher 2008) where they don't.

Finally, we show explicitly how the payoff functions in the Bayesian equilibrium of the finite game converge to the payoff functions we described for the continuum model in the first part of the paper as the number of workers and firms grows large. In particular, we show how large numbers appear to equalize the application probabilities across firms. The limit theorems make it straightforward to show that *pure strategy equilibrium* of the finite search game support allocations that converge to the equilibrium allocations described in the first part of the paper.

A few papers from the literature are worth mentioning to put the arguments here in context. Papers by (Shi 2002), and (Shimer 2005) resemble this one in that matching of worker and firm types is not assortative in equilibrium. They differ from this paper in that they allow firms to set wages that are conditional on the type of the worker who they ultimately hire. As a consequence, the logic that breaks down assortative matching is much different than it is here. The exact differences are easier to explain once the details of the model have been made clearer, so the discussion of the differences is deferred to Section 8. The papers by (Lang, Manove, and Dickens 1999) and (Eeckhout and Kircher 2008) provide directed search models that support assortative matching. In (Eeckhout and Kircher 2008) this is accomplished by having firms declare the profits they require rather than the wage they pay.[3] This has the effect of making the wage a worker receives from a firm depend on the worker's type. However, firms can't control how the wage varies with worker type, and this prevents them from adjusting wages to ensure all types will want to apply. (Lang, Manove, and Dickens 1999) is about racial discrimination, so there are only two different types of worker, which permits assortative matching. Again, we defer discussion of this point until the model in this paper has been described in more detail.

2. Fundamentals

A labor market consists of measurable sets M and N of firms and workers respectively. We normalize the measure of the set of firms to 1. The measure of the set of workers is assumed to be τ. Each worker has a characteristic y contained in a closed connected interval $Y = \left[\underline{y}, \overline{y}\right] \subset \mathbb{R}^{+}$. These characteristics are observable to firms once workers apply, but initially, they are private information to workers. Let F be a differentiable and monotonically increasing distribution

[3]Of course, they don't model it in this crude way - they consider a model in which a firm sets the price that it wants from consumers.

function. Assume that $\tau F(y)$ is the measure of workers whose type is less than or equal to y. These characteristics are assumed to be non-contractible. Firms are not able to make wages vary directly with these characteristics. A worker's payoff if he finds a job is simply the wage he receives. If he fails to find a job, his payoff is zero. Workers are risk neutral.

Firms characteristics are drawn from a set $X = [\underline{x}, \overline{x}]$. Let $H(x)$ be the measure of the set of firms whose characteristics are less than or equal to x, with $H(\overline{x}) = 1$ as described above. H is assumed integrable with convex support. Each firm has a single job that it wants to fill. It chooses the wage that it wishes to pay the worker who fills this job. Each firm's wage is chosen from a compact interval $W \subset \mathbb{R}^+$. Payoffs for firms depend on the wage they offer and on the characteristic of the worker they hire, and, of course, on their own characteristic. The payoff for every firm who hires a worker is $v : W \times Y \times X \rightarrow \mathbb{R}$ where $x \in X$ is the firm's type and $y \in Y$ is the characteristic of the worker whom she hires. It is assumed that v is continuously differentiable in all its arguments, concave in w and bounded. To maintain an order on firm types, it is assumed that for any pair (w, y) and (w', y') with $(w, y) \geq (w', y')$, if $v(w, y, x) \geq v(w', y', x)$ for some type x, then $v(w, y, x') \geq v(w', y', x')$ for any higher type $x' \geq x$. In words, this *single crossing condition* says that higher type firms assign a higher value to higher type workers. Finally, a firm who doesn't hire receives payoff 0. We use the assumption that $v(\cdot, 0, \cdot) = 0$ uniformly, which means that not hiring is treated the same way as hiring a worker with type 0.

3. The Market

Each firm in the market commits to a wage. As is typically assumed in directed search, each worker applies to one and only one firm. The firm is assumed to hire the highest type applicant who applies. It is assumed that a firm who advertises a job is committed to hire a worker as long as some worker applies even if the firm's perceived ex post payoff from the best worker who applies is negative.[4] This assumption could be defended by observing that all the applicants who apply to the firm are verifiably qualified for the job being offered. A refusal to hire *any* applicant might be problematic for legal reasons, though, of course, no firm is required to hire every qualified applicant who applies. However, the assumption is primarily intended to simplify the limit arguments

[4] A high type firm who sets a high wage in order to attract high type workers might not be willing to pay a low type worker that wage.

given in the last section of the paper. The payoff functions in the large game described in this section can be modified in a straightforward way to allow firms to decide not to hire when they have applicants.

The payoffs that players receive depend on their own actions, and on the distributions of actions taken by the other players. We specify these payoffs using standard arguments from directed search, then provide a micro foundation in Section 8. Let G be the wage offer distribution and P the joint distribution of applications, where $P(w, y)$ is understood to be the measure of the set of workers of type y or less who apply at wage w or less. We let $p_w(y)$ refer to the conditional distribution function that describes the measure of the set of applicants of type less than or equal to y who apply at wage w.

A worker of type y is always chosen over workers with lower types wherever he applies. As a consequence, he is concerned not with the total number of applicants expected to apply at the firm where he applies (the 'queue size'), but with the measure of the set of applicants whose type is as least as large as his. This number is given by

$$\int_y^{\overline{y}} dp_w(\tilde{y}).$$

We use the familiar formula $e^{-\int_y^{\overline{y}} dp_w(\tilde{y})}$ to give the probability of trade.[5] This provides worker payoffs for wages in the support of G as

$$U(w, y, G, P) \equiv we^{-\int_y^{\overline{y}} dp_w(\tilde{y})}. \tag{3.1}$$

For firms, let w be a wage in the support of G. The payoff from employing such a worker is $v(w, x, y)$. Integrating over the set of worker types gives expected profit

$$V(w, x, G, P) \equiv \int_{\underline{y}}^{\overline{y}} v(w, y, x) e^{-\int_y^{\overline{y}} dp_w(y')} dp_w(y) \tag{3.2}$$

for any wage in the support of G.

In order to describe equilibrium, we need to define payoffs for both workers and firms for wages that lie outside the support of G. In what follows, let $\overline{G}$ mean the support of G. The notation $\overline{w}$ means the highest wage in the support and $\underline{w}$ means the lowest wage in the support. We can now define the payoffs using an argument similar to (Acemoglu and Shimer 2000). Define

$$\omega(y) \equiv \max_{w \in \overline{G}} we^{-\int_y^{\overline{y}} dp_w(y')}.$$

[5]We show below that this expression is the limit value of the matching probability in a conventional urn-ball matching game.

The function $\omega(y)$ is monotonic and continuous, so it is differentiable almost everywhere. We refer to $\omega(y)$ henceforth as the *market payoff.* We assume, as do (Acemoglu and Shimer 2000) (among others), that application strategies adjust so that for any wage $w' \notin \overline{G}$ all workers whose market payoff is less than w' receive exactly their market payoff when they apply at w', while all other workers receive payoff w'. In particular, we assume $U(w', y, G, P) = \min[\omega(y), w']$. For each wage w', we use the marginal distribution $p_{w'}$that supports this property to compute the firm's profit when it offers such a wage.

To find this marginal distribution, we solve the functional equation

$$w' e^{-\int_y^{\overline{y}} dp_{w'}(y')} = \omega(y) \tag{3.3}$$

on the support of $p_{w'}$.

The solution for $p_{w'}$ depends on whether w' is above or below the support of G.[6] We discuss this briefly here to illustrate the method, and because the payoff for wages above the support is surprising. In the final part of the paper, we show that these payoffs functions are limits of payoff functions in finite versions of the game.

Begin with the case where $w' < \underline{w}$ (i.e., below the support). If $w' \leq \omega(\underline{y})$, then (3.3) has no solution for any y and $p_{w'}$should be uniformly zero. Otherwise, (3.3) implies that

$$\int_y^{y^*} dp_{w'}(y') = \log(w') - \log(\omega(y)).$$

The difference between the logs can be written as the integral of its derivative. Using this fact, then changing the variable in the integration gives

$$\int_y^{y^*} dp_{w'}(y') = \int_y^{y^*} \frac{\omega'(\tilde{y})}{\omega(\tilde{y})} d\tilde{y}. \tag{3.4}$$

The firm's payoff $V(w', x, G, P)$ is then given by substituting (3.4) into (3.2).

When w' strictly exceeds the highest wage in the support of G, (3.3) implies that

$$\int_y^{\overline{y}} dp_{w'}(y') = \log(w') - \log(\overline{w}) + \int_y^{\overline{y}} \frac{\omega'(\tilde{y})}{\omega(\tilde{y})} d\tilde{y}.$$

This means that to satisfy the market payoff condition, the distribution $p_{w'}$ must have an atom at $\overline{y}$ of size $\log(w'/\overline{w})$.

Heuristically, this means that when a firm sets a wage that is strictly higher than every other wage in the support of the distribution of wages,

[6]We leave out the case where G has a non-convex support since it is very similar.

it expects a set of applications from workers with the highest possible type. From this group, the firm will select an applicant randomly. This is just the usual matching problem in directed search. It is well known that in this case that if k is the measure of the set of applicants of the highest possible type, then each of them is offered the job with probability $\frac{1-e^{-k}}{k}$. The profit that the seller earns from this atom of applicants of the highest type is the measure of this set of applications, $\log(w'/\overline{w})$, times the probability with which each of them is awarded the job, $\frac{1-e^{-\log(w'/\overline{w})}}{\log(w'/\overline{w})}$, times the profit per applicant, $v(w', \overline{y}, x)$. So the profit associated with the wage w' is given by

(3.5)

$$V(w, x, G, P) \equiv \int_{\underline{y}}^{\overline{y}} v(w, y, x) \frac{\overline{w}}{w'} e^{-\int_y^{\overline{y}} \frac{\omega'(y')dy'}{\omega(y')}} \frac{\omega'(y)}{\omega(y)} dy + \left(1 - \frac{\overline{w}}{w'}\right) v(w', \overline{y}, x).$$

With these definitions for profits associated with wages outside the support of G, the payoff functions given by (3.1) and (3.2) now define a large game in the sense that each player's payoff depends on his own action and type as well as the distribution of actions of the other players.[7] Equilibrium is now defined in the usual way by requiring that the distribution of best replies to a distribution coincides with the distribution itself.

Definition 3.1. An *equilibrium* of this game is a pair of distributions G and P such that

(1) if $B_F = \{w : \exists x; V(w, x, G, P) \geq V(w', x, G, P)\, \forall w' \in W\}$then

$$\int_{B_F} dG(w) = 1; \tag{3.6}$$

and

(2) if $B_W = \left((w, y) : U(w, y, G, P) \geq U(w', y, G, P)\, \forall w' \in \overline{G}\right)$ then

$$\int_{B_W} dP(w, y) = \tau \tag{3.7}$$

4. Reservation Wages

We first demonstrate that equilibrium application strategies must satisfy a *reservation wage* property. Workers will effectively pick the lowest wage to which they will apply, then apply with equal probability to all firms whose wages are higher.[8]

[7] For example, see (Mas-Colell 2002), (Mas-Colell 1975) or (Schmeidler 1973).

[8] The proof of the following Proposition was suggested by a referee.

Proposition 4.1. *In any equilibrium, and for every wage in the support of G,*

$$p_w(y) = \begin{cases} \int_{\underline{y}}^{y} \frac{\tau dF(y')}{1-G^-(\omega(y'))} & \text{if } w \geq \omega(y) \\ \int_{\underline{y}}^{y^*(w)} \frac{\tau dF(y')}{1-G^-(\omega(y'))} & \text{otherwise,} \end{cases}$$

where $G^-(\omega(y)) = \lim_{x\uparrow\omega(y)} G(x)$ and $y^*(w) = \sup_y \{y' : \omega(y') \leq w\}$.

Proof. First observe that by definition, $\omega(y) \geq we^{-\int_y^{\overline{y}} dp_w(y')}$ for all w. Then if $w < \omega(y)$ there is no no-negative distribution function p_w for which (3.7) could be satisfied. So $p_w(y)$ is constant for all y for which $w < \omega(y)$. To prove the reservation wage property, we want to show that if y is in the support of p_w, then it is in the support of $p_{w'}$for all $w' > w$. To accomplish this, we show the stronger result that $w > \omega(y)$ implies that y is in the support of p_w.

Suppose $w > \omega(y)$ for some pair (w, y). Observe first that there must be some set B of F positive measure such that $y' > y$ and y' is in the support of p_w. If that weren't true, then worker y would be hired for sure at wage w, which would contradict $w > \omega(y)$. By condition (3.7) in the definition of equilibrium

$$\omega(y') = we^{-\int_{y'}^{\overline{y}} dp_w(\tilde{y})}$$

for almost all $y' \in B$.

Now suppose there is also a subset B^- which has strictly positive F-measure, contains only types $y' \geq y$ who have market payoff $\omega(y') > w$, but which is not contained in the support of p_w. Then workers whose types are in B don't compete against workers whose types are in B^- when they apply at wage w. Yet for the same reason, workers whose types are in B^- don't compete against other workers whose types are in B^- when they apply at wage w. Since almost all workers in B^- are supposed to apply at wage w with probability 0, there must be a pair of distinct worker types, say $y_0 < y_1$, such that $y_0 \in B^-$, $y_1 \in B$, and

$$we^{-\int_{y_0}^{\overline{y}} dp_w(\tilde{y})} = we^{-\int_{y_1}^{\overline{y}} dp_w(w)} \geq$$

$$\omega(y_1) > \omega(y_0).$$

Since this contradicts the definition of $\omega(y_0)$, we conclude that almost all y for which $w > \omega(y)$ are in the support of p_w.

Since p_w is absolutely continuous with respect to F, we can write

$$\omega(y) = we^{-\int_y^{\overline{y}} p(w,\tilde{y})\tau dF(\tilde{y})},$$

where $p(w,\tilde{y})\,\tau dF(\tilde{y})$ is the Radon-Nikodym derivative of p_w. Differentiating with respect to y gives

$$\omega'(y) = we^{-\int_y^{\overline{y}} p(w,\tilde{y})\tau dF(y)} p(w,y)$$

or

$$p(w,y) = \frac{\omega'(y)}{\omega(y)}$$

almost everywhere, which is independent of w. Since $\int_{\omega(y)}^{\overline{w}} p(w,y)\,dG(w) = 1$ for each y, we have $p(w,y) = \frac{1}{1-G^{-}(\omega(y))}$ which gives the result that workers apply equally to all firms whose wage is as high as their market payoff. □

We can now use Proposition 4.1 to provide a characterization of equilibrium. It implies that we can interpret the market payoff $\omega(y)$ as the lowest wage to which a worker of type y applies Then the inverse function $\omega^{-1}(w) \equiv y^*(w)$ has a natural interpretation as the highest type who applies with positive probability at a wage w.[9] When a worker of type y applies at wage w, he doesn't need to worry about workers whose types are lower than his, or workers whose types are such that their reservation wages exceed w. By Proposition 4.1, the workers whose types are between y and $y^*(w)$ are using a relatively simple application rule that has them applying with equal probability at all firms whose wage is above their reservation wage. Substituting the result in Proposition 4.1 into (3.1) gives the worker's expected payoff when he applies at wage w as

$$we^{-\int_y^{y^*(w)} k(y')dF(y')}, \tag{4.1}$$

where we substitute

$$k(y) \equiv \frac{\tau}{1-G^{-}(\omega(y))}$$

in order to simplify the formula slightly.

Firms' payoffs can be similiarly simplified. A firm who offers wage w in the support of G will attract workers whose types are between $\underline{y}$ and $y^*(w)$. Each such worker is hired conditional on applying with probability $e^{-\int_y^{y^*(w)} k(y')dF(y')}$ as just described. The probability that such a worker applies at wage w is $1 - G^{-}(\omega(y))$, which gives the expected revenue to a firm of type x from a worker of type y as

$$\frac{v(w,y,x)e^{-\int_y^{y^*(w)} k(y')dF(y')}}{1-G^{-}(\omega(y))}.$$

[9] In case ω is not monotonic, use $\omega^{-1}(w) = \sup_y \{y' : \omega(y') \le w\}$.

Adding this up over all the worker types who apply with positive probability gives the firm's profit function

$$\int_{\underline{y}}^{y^*(w)} v(w,y,x) \frac{e^{-\int_y^{y^*(w)} k(y')dF(y')}}{1-G^-(\omega(y))} \tau f(y)dy =$$

$$\int_{\underline{y}}^{y^*(w)} k(y)v(w,y,x)e^{-\int_y^{y^*(w)} k(y')dF(y')}dF(y) \tag{4.2}$$

for each wage in the support of G.

To simplify the case where $w' < \underline{w}$, begin with type $y^*(\underline{w})$ which is the highest type of worker who will apply at wage $\underline{w}$. Each worker whose type is below $y^*(\underline{w})$ has a market value below $\underline{w}$. The measure of the set of firms whose wage is at least as high as their reservation wage is then 1, and $k(y)$ is simply τ. So this market value is given by

$$v(y) = \underline{w}e^{-\int_y^{y^*(w_0)} \tau dF(y')}.$$

Then from (3.3), the probability that worker y is hired at wage $\underline{w}$ must be equal to

$$\frac{\underline{w}e^{-\int_y^{y^*(w_0)} \tau dF(y')}}{w'}.$$

This simple multiplication makes all workers whose types are below $y^*(\underline{w})$ indifferent between the deviator and the lowest wage in the support of G.

Using this matching probability gives the firm's profit when it offers w' as

$$\int_{\underline{y}}^{y(w')} \tau v(w',y,x)\frac{\underline{w}}{w}e^{-\int_y^{y^*(\underline{w})} \tau dF(y')}F'(y)dy. \tag{4.3}$$

Finally, substituting the results of Proposition 4.1 into (3.5) gives the profits for a firm who offers a wage that is strictly higher than any wage in the support of G as

(4.4)

$$\frac{\overline{w}}{w'}\int_{\underline{y}}^{\overline{y}} k(y)v(w',y,x)e^{-\int_y^{\overline{y}} k(y')dF(y')}F'(y)dy + v(w',\overline{y},x)\left(1-\frac{\overline{w}}{w'}\right).$$

5. Equilibrium

We can now present the main characterization theorem.

Proposition 5.1. *Suppose that both the function $\iota(w,y,x) \equiv \frac{v(w,y,x)}{w}$ and its derivative with respect to w are non-decreasing in x. Then a pair (G,P) is an equilibrium if and only if there is a point y_0 and a pair* *of functions* $\omega(y)$ *and* $h(y)$ *satisfying* $\omega(y_0) = \underline{w}$*, and* $G^{-}(\omega(y)) = H(h(y))$*, such that*

$$\omega(y)\frac{\tau}{1-H(h(y))}F'(y) = \omega'(y)\,; \tag{5.1}$$

and

$$v\left[\omega(y), y, h(y)\right] = -\int_{\underline{y}}^{y}\left[v_w\left(\omega(y), y', h(y)\right) - \frac{v\left(\omega(y), y', h(y)\right)}{\omega(y)}\right]\omega'(y')\,dy'. \tag{5.2}$$

This Proposition characterizes the equilibrium in a manner that is familiar in the directed search literature. The function ω is the market payoff function of workers, or the market utility function. The worker of type y_0 breaks the set of worker types into two parts. Workers whose types are at or above y_0 have some wage where they can apply and be hired for sure. For them, the function $\omega(y)$ represents their reservation wage. The worker of type y_0 in particular is sure to be hired if he applies at the lowest wage $\underline{w}$ offered by any firm. Worker types below y_0 have a chance of losing out on the job even if they apply at the lowest wage $\underline{w}$. The function $h(y)$ identifies the type of the firm who offers worker y's reservation wage when $y \geq y_0$, while $h(y) = \underline{x}$ if $y < y_0$.

The two conditions can be interpreted in the usual way as tangency conditions. For example, the payoff when a worker applies to a firm is a function of the wage that the firm offers and the highest worker type who applies that firm. Each worker type should attain a payoff that maximizes this function across all wage-highest type pairs that provide the market payoff to some worker type. Equation (5.1) then expresses the fact that a worker of type y has an indifference curve that is tangent to the market payoff function ω at y.

Similarly, interpret the firm's profit function (4.2) as a function of the wage that it pays and the highest worker type it attracts. The firm's problem is then to choose a wage and highest worker type that maximizes its profit conditional on providing some worker type his market payoff. The equation (5.2) expresses the requirement that an iso-profit curve for a firm of type $h(y)$ is tangent to the market payoff function $\omega(y)$ at the point $(\omega(y), y)$.

It may be worth noting here that despite the tangency, this equilibrium won't be efficient. The reason is that this allocation doesn't do a good job matching worker and firm types. To see this, observe that from (Shimer 2005), an efficient allocation is attained when firms can pay workers a wage that depends on their types. Generally, a low type worker who applies to a high type firm in Shimer's model is rewarded with a lower wage than that paid to the high type workers. This focuses low type applications at low type firms, which limits mismatching. Here a low type worker is provided the same wage as a high type worker, providing the low type worker a much bigger incentive to apply at high wage firms. It is this concentration of applications with the high type firms which precludes efficiency.

Proof. Start with an equilibrium pair P and G. Let $\underline{w}$ be the lowest wage in the support of G and $y_0 = \sup\{y : \omega(y) \leq \underline{w}\}$. From Proposition 4.1, each worker type y must attain his market payoff $\omega(y)$ by applying at any wage above his reserve price. So

$$we^{-\int_y^{y^*(w)} k(y')dF(y')} = \text{constant} \tag{5.3}$$

for each $w \geq \omega(y)$. For each worker type y, the derivative of this expression with respect to w should then be zero at every wage above $\omega(y)$. That is

$$we^{-\int_y^{y^*(w)} k(y')dF(y')} k(y^*(w))F'(y^*(w))\frac{dy^*(w)}{dw} = e^{-\int_y^{y^*(w)} k(y')dF(y')}$$

giving

$$w\frac{\tau}{1-G^-(w)}F'(y^*(w)) = \frac{1}{dy^*(w)/dw} \tag{5.4}$$

Since $y^*(w)$ is the inverse function of $\omega(y)$ at each point at which ω is increasing,

$$\omega(y)\frac{\tau}{1-G^-(\omega(y))}F'(y) = \frac{d\omega(y)}{dy} \tag{5.5}$$

at each $y \geq y_0$. Fix $h(y)$ to be the solution to $G^-(\omega(y)) = H(h(y))$. The result (5.1) then follows from the definition of h.

Now use the market payoff function defined by (5.1) and its extension to types below y_0 to simplify the firm's profit function. The firm's profit function when it sets wage $w \geq \omega(y_0)$ is given by

$$V(w, x, G, \omega) = \int_{\underline{y}}^{y^*(w)} k(y)v(w, y, x)\frac{\omega(y)}{w}F'(y)dy.$$

Now substituting (5.1) into the firm's profit function gives

$$V(w, x, G, \omega) = \int_{\underline{y}}^{y^*(w)} v(w, y, x)\frac{\omega'(y)}{w}dy$$

Observe that choosing w, then figuring out what worker types will apply by finding the worker type who has reservation wage w is equivalent to choosing the highest worker type that will apply, then setting the wage equal to that worker type's reservation wage. Firm profits can then be written as functions of the highest worker type who applies as

$$V\left(\omega\left(y\right),x,G,\omega\right)=\int_{\underline{y}}^{y}v(\omega\left(y\right),y',x)\frac{\omega'\left(y'\right)}{\omega\left(y\right)}dy' \tag{5.6}$$

Maximizing (5.6) with respect to y gives the first order condition

$$v\left[\omega\left(y\right),y,x\right]=-\int_{\underline{y}}^{y}\left[v_w\left(\omega\left(y\right),y',x\right)-\frac{v\left(\omega\left(y\right),y',x\right)}{\omega\left(y\right)}\right]\omega'\left(y'\right)dy'$$

When G is an equilibrium, the firm who offers a wage equal to worker y's reservation wage is using a best reply. Hence this condition must hold when evaluated at $x=h\left(y\right)$ for each $y\geq y$, and this gives (5.2).

To make the argument in the other direction, observe that a solution to (5.1) holds worker payoff constant as required by the first condition of equilibrium, provide the wage distribution is given by $G\left(\omega\left(y\right)\right)=H\left(h\left(y\right)\right)$. Hence (3.7) holds for the density $p\left(w,y\right)=\frac{1}{1-G^{-}\left(w\right)}$. The aggregate distribution P can then be constructed by integrating this density.

From equation (4.4), it is straightforward to verify that local profit maximization conditions hold at $\omega\left(\overline{y}\right)$ whenever (5.2) holds. It is then straightforward to verify by brute force that second order conditions are guaranteed by the assumption that $v\left(w,y,x\right)$ is concave in w for every y. The same approach verifies the second order condition at w_0.

Inside the support, the condition (5.2) guarantees that the first order conditions for profit maximization hold when every firm of type $h\left(y\right)$ offers wage $\omega\left(y\right)$. If the second order condition fails, then the iso-profit curve in $\left(w,y\right)$ space for firm $h\left(y\right)$ must cross ω at another wage. The argument is similar whether the wage at this crossing point is higher or lower than $\omega\left(y\right)$, so suppose the crossing occurs at a higher wage. Then, firm $h\left(y\right)$'s iso-profit curve is strictly steeper than ω at this higher wage w'. Yet by (5.2) there is some higher type firm $x'>h\left(y\right)$ whose iso-profit curve is tangent to ω at w'. Since the slope of the iso-profit curve is

$$-\frac{\iota\left(w,y,x\right)}{\int_{\underline{y}}^{y}\iota_w\left(w,y',x\right)\omega'\left(y'\right)dy'}$$

and this ratio is non-decreasing in x by assumption, this leads to a contradiction. □

In this Theorem, y_0 is the highest worker type who applies to the lowest wage in the support of the equilibrium wage distribution. Condition (5.1) ensures that payoffs are constant above at all wages above the reservation wage. The advantage of the exponential matching function is that a reservation wage rule that satisfies this single functional equation ensures the constant payoff condition for all worker types. The condition (5.2) is the first order condition for profit maximization - i.e., $V_w(w, x, G, \omega) = 0$. The term on the left hand side of the condition is the marginal gain associated with attracting a higher type when wage is increased. The term on the right hand side is the marginal cost of paying a higher wage to all the lower types.

The functional $h(y)$ has a natural interpretation as the lowest type of the firm who offers the reservation wage of worker y. From the type y_0 and the functional equations ω and h, the wage distribution is readily constructed. The lowest wage in the support of the distribution is $\omega(y_0)$. Then for each $y > y_0$, $G^-(\omega(y)) = H(h(y))$.

6. Examples with Identical Firms

To begin, we will assume that all firms have the same profit function. The reason this case is amenable to analysis is because the first order condition (5.2) reduces to a constant profit condition that involves only the function ω, and a boundary condition. If we begin with a function ω that satisfies this constant profit condition, we can then find the function G that ensures that (5.1) is satisfied. If this function is a distribution function, then we have an equilibrium with a non-degenerate wage distribution.

Example 6.1. If $v(w, y, x) = 1 + \alpha y - w$ for all $x \in [\underline{x}, \overline{x}]$, then there is a degenerate single wage equilibrium with $\underline{w} = 1 + \alpha\overline{y} - \int_{\underline{y}}^{\overline{y}} (1 + \alpha y')\, de^{-\int_{y'}^{\overline{y}} \tau F'(t)dt}$.

Proof. Proposition 5.1 applies to the degenerate case since we can set $y_0 = \overline{y}$ and $h(\overline{y}) = \underline{x}$. To see this, observe that if the wage distribution is degenerate at some wage w_0, then all workers apply at this wage. Since y_0 is the highest type who applies to the lowest wage in the support, the conclusion $y_0 = \overline{y}$ follows. The function $h(\overline{y})$ is supposed to be the lowest firm type who offers worker $\overline{y}$ his reservation wage. Since all firms offer the same wage, this is $\underline{x}$. We then have trivially from (5.1) $\omega(\overline{y})\tau F'(\overline{y}) = \underline{w}\tau F'(\overline{y}) = \omega'(\overline{y})$.

The first order condition (5.2) then becomes (since $G^-(\omega(y)) = G^-(\underline{w}) = 0$)

$$v[\underline{w}, \overline{y}, \underline{x}] = -\int_{\underline{y}}^{\overline{y}} \left[v_w(\underline{w}, y', \underline{x}) - \frac{v(\underline{w}, y', \underline{x})}{\underline{w}} \right] \omega'(y')\, dy'.$$

Substituting the specific profit function $1+\alpha y-w$ then gives the first order condition

$$1+\alpha\overline{y}-\underline{w}=\int_{\underline{y}}^{\overline{y}}\left(1+\alpha y'\right)de^{-\int_{y'}^{\overline{y}}\tau F'(t)dt} \tag{6.1}$$

which gives the result. □

This equilibrium looks very much like any other generic directed search equilibrium in that a common wage is chosen so that no firm has any incentive to increase its wage offer. (Lang and Manove 2003) prove a similar single wage result with a continuum of worker types under the assumption that firms' profits are independent of worker types. Their model is a special case of this example in which $\alpha=0$. Substituting $\alpha=0$ into (6.1) and solving gives $\underline{w}=e^{-\tau}$. The result is derived here by simplifying the functional equations in Proposition 5.1. To see the argument in a more conventional way[10], note that since firms don't care who they hire, they simply want to maximize expected profit. These expected profits are given by $1-w$ times the probability that at least one worker applies, say $1-e^{-l}$. Since firms always choose the highest type worker who applies, this means that e^{-l} is the probability that the *lowest* type worker is hired if he applies to the firm. Since all the firm cares about is the trading probability, the equilibrium can be described using the usual market utility story - the firm should choose a wage to maximize $(1-w)\left(1-e^{-l}\right)$ subject to the constraint that $we^{-l}=\underline{w}e^{-\tau}$, where $\underline{w}e^{-\tau}$ represents the expected payoff to the lowest worker type when he applies to one of the other firms. It is straightforward to check that $\underline{w}$ can only be a solution to this problem if $\underline{w}=e^{-\tau}$ as above. When firms care about worker types directly, the argument is more subtle since firms also care about the highest type who applies at any wage. It is this feature that is captured by (5.2) in Proposition 5.1.

Example 6.2. Suppose that $v(w,y,x)=(\alpha y-w)$ for all x. Then a wage distribution can be supported in equilibrium only if $F'(y)$ is decreasing.

Proof. Fix the lowest wage $\underline{w}$ in the support of the equilibrium distribution and let y_0 be the type for whom $\omega(y_0)=\underline{w}$. When firms have the same profit function, all wages in the support must yield the same profit. This is guaranteed by condition (5.2), which after substituting

[10] I am grateful to a referee for pointing this out.

the special profit function becomes

$$\alpha y - \omega(y) = -\int_{\underline{y}}^{y} \left[1 - \frac{\alpha y' - \omega(y)}{\omega(y)}\right] \omega'(y')\, dy'.$$

Rewriting slightly gives

$$(\alpha y - \omega(y))\, \omega(y) = \int_{\underline{y}}^{y} \alpha y' \omega'(y')\, dy'$$

Since this must hold uniformly in y, the derivatives of this expression with respect to y must also be the same, or

$$(\alpha y - \omega(y))\, \omega'(y) + \omega(y)\,(\alpha - \omega'(y)) = \alpha y \omega'(y).$$

This gives the simple condition $\omega'(y) = \frac{\alpha}{2}$. This is the condition that the market payoff function must have when $\omega(y)$ is in the support of the equilibrium wage distribution in order for firms profits to be constant on the support of this distribution. From Condition (5.1) it must be that

$$\omega(y)\, \frac{\tau F'(y)}{1 - G(\omega(y))} = \frac{\alpha}{2}$$

along the support of the equilibrium wage distribution. Since $\frac{\omega(y)}{1-G(\omega(y))}$ is strictly increasing in y, this condition can't be fulfilled unless $F'(y)$ is decreasing. $\square$

One implication of this result is that if worker types are uniformly distributed, then the only equilibrium that can supported with identical firms has all firms offering the same wage.

Example 6.3. Suppose that $v(w, x, y) = y - w$, and that $F(y) = y(2-y)$ with $\underline{y} = 0$ and $\overline{y} = 1$. Then there is a worker firm ratio $\tau_0 < \frac{3}{2}$ such that a non-degenerate distribution of wages can be supported in equilibrium for the economy where the ratio of workers to firms is τ_0. The equilibrium wage distribution is convex and has support $\left[\frac{1}{2\tau_0}, \frac{1}{2\tau_0} + \frac{1}{4}\right]$.

The proof of these assertions are in the appendix. This example illustrates a difference between this paper and (Lang and Manove 2003) for which only single wage equilibrium exist with identical firms. The difference is that in this latter reference, firms don't care what worker they hire. The differences in matching probabilities associated with higher wages won't in itself support a distribution of wages. Here, higher wages can also bring improvements in worker quality which is what supports the distribution here.

7. Offer Distributions and Duration

These examples illustrate how the functional equations can be used to analyze equilibrium. They illustrate that equilibrium might not be unique. With identical firms, non-degenerate wage distributions may or may not exist in equilibrium, depending both on equilibrium selection and primitive. In this section we return to the case where firms differ, so that equilibrium wage offer distributions will generally be non-degenerate. In particular, the focus here is on the relationship between employment probability and wage. Employment probabilities at different wages are unobservable. However, some insight into this can be gleaned from unemployment duration. For this section, we imagine the equilibrium wage distribution G to be a steady state distribution associated with a repeated version of the model in which a worker of type y who fails to find a job goes into the next period with the same type, faces the same wage and worker type distribution, so plays the same mixed strategy again in the next period. To do this properly we should model all this dynamics. However, for the interpretive results in this section, this informal interpretation should be sufficient.

If the probability $Q(y)$ with which a worker finds a job is the same in each period when he is unemployed, then the average number of periods that will elapse before he finds a job is just $\frac{1}{Q(y)}$. This observation makes it possible to work out the relationship between duration of unemployment and the wage at which a worker leaves unemployment. Suppose this duration function is $\Phi(w)$ - i.e., $\Phi(w)$ is the average unemployment duration for workers who find jobs at wage w.

Since workers apply with equal probability at all openings where the wage is above their reservation wage, it is possible to work out the probability that a worker of type y is hired by *some* firm. There are $G'(w)$ firms who offer wage w and $1 - G(\omega(y))$ firms who offer wages above the worker's reservation wage. So the 'probability' that a worker applies at wage w is $\frac{G'(w)}{1-G(\omega(y))}$. The probability of matching with such a firm is $e^{-\int_y^{y^*(w)} k(y')dF(y')}$, so the probability that a worker matches if he follows his equilibrium application strategy is

$$Q(y) = \int_{\omega(y)}^{\overline{w}_G} e^{-\int_y^{y^*(w)} k(y')dF(y')} \frac{G'(w)}{1 - G(\omega(y))} dw$$

where $\overline{w}_G$ is the highest wage in the support of G.[11] Since the expected wage is constant for a worker of type y at every wage above $\omega(y)$, this

[11]In this expression the function $\omega(y)$ has to be interpreted as the lowest wage to which a worker of type y applies instead of as his market payoff. This two things

can be written as

$$Q(y) = \int_{\omega(y)}^{\overline{w}_G} \frac{\omega(y)}{w} \frac{G'(w)}{1 - G(\omega(y))} dw.$$

It isn't immediate that better workers will have higher matching probabilities or lower durations. The reason is that higher types have higher reservation wages. So despite the fact that they are more likely to be hired at any particular firm than low type workers, they tend to apply at firms where there is a lot of high type competition. From the last equation and the fact that workers' reservation wages increase in their types, employment probability will be an increasing function of type if

$$\psi(w) = \int_{w}^{\overline{w}_G} \frac{w}{w'} \frac{G'(w')}{1 - G(w)} dw'$$

is an increasing function of the wage w.[12][13]

This function represents the expectation of the ratio of any wage to the harmonic mean of higher wages in the distribution G. This function isn't particularly simple conceptually. Nor is it easy to deduce distributions for the unobservables that will support this property. However, it is relatively easy to check. For example, it is straightforward to check that the equilibrium distribution given in closed form in Example 6.3 has the property that this function is increasing. We explain below how to check this condition using the accepted wage distribution (which is easier to observe).

Workers types can't be observed. What is observable is the actual duration of workers hired at different wages. To establish the final connection we simply have to show that firms that set high wages and hire a worker actually end up with better workers i.e., workers with higher matching probabilities. The probability $\tilde{F}(y_0|w)$ that a worker hired by the firm who offers wage w has a type less than or equal to y_0 is given by

$$\tilde{F}(y_0|w) = \frac{\int_{\underline{y}}^{y_0} k(y) e^{-\int_y^{y^*(w)} k(y')dF(y')} F'(y)dy}{\int_{\underline{y}}^{y^*(w)} k(y) e^{-\int_y^{y^*(w)} k(y')dF(y')} F'(y)dy} =$$

Note that this probability is conditional on some worker being hired by the firm, which explains the denominator. Substituting for $k(y)$ and

can differ for a worker whose type is low enough that there is no wage at which he will be surely hired.

[12]See the previous footnote to see why the lower bound of the integration is w.

[13]Notice that the function ψ must be increasing somewhere. It is obviously less than 1 when w is at the bottom of the support, and equal to 1 at $\overline{w}_G$.

using (5.3), and (5.5) gives an even simpler formulation

$$\tilde{F}\left(y_0|w\right) = \frac{\int_{\underline{y}}^{y_0} \omega'(y)dy}{w - \omega(\underline{y})} \tag{7.1}$$

This expression is readily seen to be declining in w. The interpretation is that an increase in the wage moves the distribution function for the type hired by the firm to one that first order stochastically dominates the original distribution.

The theorem that follows is a consequence of computing the duration as $\Phi(w) = \frac{1}{\int Q(y)d\tilde{F}(y|w)}$.

Proposition 7.1. *If $\psi(w)$ is monotonically increasing then the expected duration of unemployment for a worker hired by a firm is an decreasing function of the wage offered by the firm.*

When $\psi(w)$ is increasing, workers who are hired at high wage firms will tend on average to have spent less time searching for jobs than workers who are hired by low wage firms. This is quite unlike standard directed search where high wages and long duration must go together. This prediction is not a particularly strong test of the model, since the function $\psi(w)$ may not be monotonic. Notice however, that it is a testable consequence of the model that does not rely on any knowledge about the distributions of the unobservables.

The expected duration for a worker hired by a firm offering a wage w is given by the reciprocal of

$$\int_{\underline{y}}^{y^*(w)} \frac{\omega'(y)}{w - \omega(\underline{y})} \psi(\omega(y))dy$$

using the expression for the density of the type of worker hired by the firm that was derived above. It is apparent from this expression that when $\psi(w)$ is non-monotonic, then there will be no systematic relationship between the wage at which a worker is hired and his probability of matching measured as his expected duration. Even in this dimension, the result is quite different from the standard directed search model where wage and employment probability must be inversely related.

Finally, whether duration and wage are inversely related or not, a simple change of variable in the expression $\int Q\left(y\right) d\tilde{F}\left(y|w\right)$ gives the following result:[14]

[14] I am grateful to Vadim Marmer for pointing out this connection.

Proposition 7.2. *In every equilibrium of the large directed search game*

$$\frac{1}{w - \underline{w}} \int_{\underline{w}}^{w} \psi\left(w\right) dw = \frac{1}{\Phi(w)}.$$

The expression on the right is the expected duration function, the expression on the left is the condition derived from the equilibrium conditions. The function ψ is a simple function of the wage offer distribution, while $\Phi\left(w\right)$ is the observed relationship between duration and exit wage.

One problem with these results is that they are based on the wage offer distribution, which is not observable. The relationship between the wage offer distribution and the accepted wage distribution is relatively straightforward since the accepted wage distribution can be computed from the wage offer distribution using the equilibrium application strategy and probabilities of being hired.

Let G^* be the observed (or accepted wage distribution). Then

Proposition 7.3. *The wage offer distribution G and the accepted wage distribution are related by*

$$G\left(w\right) = G^*\left(w\right) + \omega\left(\underline{y}\right) \int_{\underline{w}}^{w} \frac{G^{*\prime}\left(w'\right)}{w'\left(1 - \frac{\omega\left(\underline{y}\right)}{w'}\right)} dw'.$$

The proof of this Proposition is in the online appendix. On the right hand side of this expression, the only unobservable is the payoff to the lowest type worker $\omega\left(\underline{y}\right)$. This could be estimated from survey data by using the worst observed experience, or possibly by using an outside option like unemployment insurance to define the lowest attainable payoff.[15] Up to this identifying assumption, this formula can be used to convert the results of Propositions 7.1 and 7.2 into statements about the observable accepted wage distribution.

8. Equilibrium of the Worker Application Sub-game

Assertions about payoffs in large games are ultimately ad hoc. In this section we analyze a finite game to show how the payoffs in the large game come about. The finite game also makes it possible to illustrate how the model discussed above differs from some of the other well known papers involving different worker types. In particular in the

[15] The lowest observed wage will typically exceed the payoff of the lowest worker type if the worst workers type is so low that he won't hired for sure even at the lowest wage.

finite model it is easy to see how workers mixed application strategies differ from the mixing that occurs in models like (Shimer 2005) and (Shi 2002) where wages can be conditioned on worker type. Furthermore the distinction between two type models like (Lang, Manove, and Dickens 1999) and continuous type models can be made clear. Finally, the finite type model illustrates why assortative matching of the sort that occurs in (Eeckhout and Kircher 2008) cannot occur here.

For this section, there are n workers and m firms. Each worker's type is an independent draw from some common distribution F. Worker and firm payoffs are as described above. Firm types are assumed to be common knowledge. Firms set wages, then workers apply. Finally firms hire the best worker who applies. The solution concept is perfect Bayesian equilibrium.

To begin, focus on the second part of the process in which workers make their applications. A strategy for worker i in the application sub-game is a function $\pi^i : W^m \times Y \to S^{m-1}$, where $S^{m-1} = \left\{\pi \in \mathbb{R}_+^m : \sum_{i=1}^m \pi_i = 1\right\}$.[16] This section analyzes symmetric equilibria in which every worker uses an application strategy that is a common function of his or her type. The idea that is fundamental to directed search is that these application strategies depend on the array of wages on offer. For the purposes of characterizing the equilibrium in the application sub-game associated with a fixed set of wages, the notation that captures this will be suppressed and we write $\pi_j(y)$ to be the probability with which each worker whose type is y applies to firm j.

Since firms always hire the worker with the highest type who applies, worker i will match with firm j in equilibrium so long as every other worker in the market either has a lower type than he does, or applies to some other firm. To calculate this, suppose worker i's type is y. The probability that some other worker has type $y' > y$ *and* chooses to come to firm j is $\pi_j\left(y'\right) dF\left(y'\right)$. So the probability that this worker will come and take the job away from worker i is $\int_y^{\overline{y}} \pi_j\left(y'\right) dF\left(y'\right)$. The probability that no other worker comes and takes the job away is

$$q\left(y, w_j\right) = \left[1 - \int_y^{\overline{y}} \pi_j\left(y'\right) dF\left(y'\right)\right]^{n-1}. \tag{8.1}$$

So $q\left(y, w_j\right)$ is the probability that worker i gets the job at wage w_j. His expected payoff when he applies to firm j is $q\left(y, w_j\right)$ multiplied by the wage w_j that the firm offers.

[16] We ignore the possibility that a worker might not apply to any firm since that is a strictly dominated strategy given the assumptions about payoffs.

This logic can also be used to derive the firm's profit function. The firm hires the best type who applies. So the firm's expected profits are determined by the probability distribution of the highest type who applies. Fix a type y. The probability that any particular worker either has a type below y or applies at some other firm, using the logic above, is $1-\int_y^{\overline{y}} \pi_j(y')\,dF(y')$. The probability that *all* the workers either have types below y or apply to another firm is

$$\left[1-\int_y^{\overline{y}} \pi_j(y')\,dF(y')\right]^n.$$

To say this a different way, this is the probability that the highest type who applies to firm j is less than or equal to y. The probability distribution function has a density given by $nq(y,w_j)\,\pi_j(y)\,F'(y)$. Integrating over possible values for this highest type gives the expected payoff function for firm j as

$$\rho(w_j,x)=\int_{\underline{y}}^{\overline{y}} v(w_j,y,x)nq(y,w_j)\,\pi_j(y)\,dF(y).$$

Notice that the functions π and q implicitly depend on the wages offered by all the firms, so these payoff functions describe a game in the usual way.

It will simplify the argument in this section to assume that wages are ordered in such a way that $w_1 \leqslant w_2 \leqslant \ldots \leqslant w_m$. In a slight abuse of notation, refer to an array $\{y_K,\ldots,y_m\}$ with $y_K \leq y_{K+1} \leq \ldots \leq y_m$ as a *partition* of the set of types. The collection of intervals $[y_k, y_{k+1})$ along with $[\underline{y}, y_K)$ constitute the sets in this partition.

The unique (symmetric) equilibrium for the application sub-game is given by the following Lemma.

Lemma 8.1. *For any array of wages $w_1,\ldots w_m$ offered by firms for which $w_1>0$, there is a partition $\{y_K,\ldots,y_m\}$ containing no more than m intervals, and a set $\{\pi_j^k\}_{k\geqslant K; j\geqslant k}$ of probabilities satisfying $\pi_j^k>0$ and $\sum_{j=k}^m \pi_j^k=1$ for each k and such that the strategy*

$$\pi_j(y)=\begin{cases}\pi_j^k & \textit{if } j\geqslant k;\, y\in[y_k,y_{k+1})\\ 0 & \textit{otherwise}\end{cases}$$

is almost everywhere a unique (symmetric) continuation equilibrium application strategy. The probabilities π_j^i satisfy

$$\left(\frac{\pi_j^i}{\pi_i^i}\right)^{n-1}=\frac{w_i}{w_j} \tag{8.2}$$

for each $j>i$.

Furthermore, the numbers $\{y_k\}$ and $\{\pi_j^k\}$ depend continuously on the wages offered by firms.

There are many indices to keep track of, but the logic is simple enough. Suppose there are only two firms offering wages $w_1 < w_2$. The highest worker types apply only to firm 2. If y is close enough to $\overline{y}$, then even if the other worker is expected to apply at wage w_2 for sure, the first worker will get the job with probability $F(y)$. If $F(y)\,w_2 > w_1$ then there is no point applying at wage w_1. This immediately describes the cutoff point $y_m = y_2$ to be the point where $F(y_2)\,w_2 = w_1$. This gives the constant $\pi_2^2 = 1$.

The main content of the theorem comes in the description of what happens to the types below y_2. The theorem says that all types below y_2 use exactly the same application probabilities π_1^1 and π_2^1 which are readily derived from the conditions

$$\pi_2^1 = \frac{w_1}{w_2}\pi_1^1$$

and the fact that π_1^1 and π_2^1 must sum to zero ($\pi_2^1 = \frac{w_1}{w_1+w_2}$). The two application strategies are then

$$\pi_1(y) = \begin{cases} \frac{w_2}{w_1+w_2} & y < y_2, \\ 0 & \text{otherwise}; \end{cases}$$

and

$$\pi_2(y) = \begin{cases} \frac{w_1}{w_1+w_2} & y < y_2 \\ 1 & \text{otherwise}. \end{cases}$$

The complete proof is included in the appendix. The theorem is hard to state because each worker type has to assign a different probability of applying to every different wage. The Lemma shows that Bayesian equilibrium puts three kinds of structure on these strategies. First, it says that if a worker sends an application with positive probability to a firm offering a wage w_k, then he or she must send applications to every higher wage as well. This is the 'reservation wage' part of the story, since the worker has to decide what is the lowest wage to which he will send an application. Second, the formula (8.2) along with the requirement that the application probabilities sum to one, then determines the entire application strategy from the reservation wage. Third, and critically important for most of the technical results in the paper, the Lemma partitions the worker type space into intervals, then says that all workers whose type is in the same interval will use exactly the same application probabilities.

To see why mixing has to occur, start with the elite types in the interval $[y_m, \overline{y}]$. As in the two firm example described above, they apply only to the highest wage firm. They might as well, since their types are so high they are very likely to be hired at every wage no matter what the other workers do. The 'marginal' worker type in this elite group has type y_m. He is just indifferent between applying at the highest wage firm and getting the job if it happens to be the case that there are no other elite workers, and applying at the second highest wage and getting that job for sure.

Now consider a worker whose type is just slightly below y_m. If the *only* workers who apply to the highest wage firm are workers in the elite group, then this infra marginal type has the same chance of finding a job with the highest wage firm as does the worker of type y_m - he will get the job if none of the other workers has an elite type. Yet if he applies at the second highest wage, there is always a chance that he will lose out to a worker whose type is in between his type and y_m. Since the worker of type y_m is indifferent between the highest and second highest wage, the worker with the lower type must strictly prefer to apply at the highest wage. So to support an equilibrium, these infra marginal workers whose types are slightly below y_m must face the same competition from other infra marginal workers at the highest wage firm as they do at the second highest wage firm. In other words, infra marginal workers must apply with positive probability at the highest wage firm. Exactly this same logic extends down through all the lower types.

This explains the difference between models with a continuum of types and models like (Lang, Manove, and Dickens 1999) where there are only two types.[17] As explained above, it is the infra marginal workers who break up any potential equilibrium where workers sort by type. When there are only two types, as there are in (Lang, Manove, and Dickens 1999), there are no infra marginal types, so complete sorting can be supported when the lower type is just indifferent between applying at the low wage and the high wage. The addition of a third worker type in their model without a corresponding firm type to hire it, would lead to an equilibrium in the application subgame that more closely resembles the equilibrium described here.

The uniqueness of the mixed equilibrium also explains why assortative matching cannot be supported in equilibrium, as it is in (Eeckhout and Kircher 2008). The difference between the two models is that

[17]Two types is a very natural assumption for their problem.

(Eeckhout and Kircher 2008) assume that the 'proposer' (or wage setter here) doesn't care directly about the types of the parties involved in a transaction. This same property could be accomplished here by changing the offer that the firm makes from a wage to a demand for profit - i.e., whoever it hires the firm requires the same profit. The firm no longer cares which worker it hires, and workers will naturally match assortatively (under the payoff restrictions that they provide). Their approach is similar to the (Shimer 2005)-(Shi 2002) approach which makes the wage contingent on worker type. Yet feasible contracts in (Eeckhout and Kircher 2008) are more restricted than in (Shimer 2005)-(Shi 2002) since firms cannot vary wages arbitrarily across types.

Models like (Shimer 2005) or (Shi 2002) where firm can condition wages on type also support mixed application strategies, but for a very different reasons. As in (Shi 2002) suppose there are only two types of firms and two types of workers. High type firms offer high wages to attract high type workers. However, by the nature of directed search, there is a chance they will nonetheless end up without applications. Firms can raise their profits by setting a *higher* wage in order to attract lower quality applicants, who they will only hire in the event that no high quality applicants apply. The higher wage for lower quality is needed to compensate the low quality workers for the low chance they will be hired. So mixing among firms of different wages is used to support the equilibrium in the wage setting part of the game.

9. Limit Payoffs

Finally, we show the sense in which the large game payoffs we described in the continuum model in the first part of the paper are limits of payoffs in finite games. There are a couple of reasons for doing this. First, the matching probabilities and payoff functions used in the continuum model are subtly different than those used in the conventional directed search literature. For instance, matching probabilities aren't based on the queue size at the firm in the usual way. Furthermore, payoffs associated with wage offers that exceed all wages in the support of the existing distribution of wages involve outcomes in which firms sometimes choose the best applicant, and sometimes select randomly from a group of highest quality applicants. Whether or not these functions seem plausible descriptions of payoff, they are essentially ad hoc. The limit theorem provided here justifies these payoff functions.

Furthermore, the symmetric continuation equilibrium in worker applications strategies is unique, this limit theorem shows not only that the heuristic descriptions of payoffs given in the continuum model are reasonable approximations to payoffs in large finite games, it also shows that these payoff functions are the only ones that can be used to approximate payoffs in large finite versions of the game.

Lastly, the limit results are designed to provide payoffs for all distributions of wages, not just those associated with equilibrium. So they do not directly address sequences of equilibrium in the wage setting part of the game. However, it is straightforward to use these results to show that every sequence of pure strategy equilibrium in finite versions of the game converges to an equilibrium in the continuum game as we have described it above. All this requires is a restriction on firms' payoff functions that ensure that they don't vary in 'unreasonable' ways with worker types.[18] The proof is by contradiction. If convergence fails, then the limit distribution of wages will have the property that a measurable set of firms will want to deviate. Then, provided payoffs aren't too sensitive to type and wage distributions, then firms will want to deviate in large finite versions of the game as well. The details of this straightforward but lengthy argument are left out in the interest of brevity.

The theorem:

Theorem 9.1. *Let G be a distribution of wages, w a wage in the support of G offered by a firm of type x. Let G_n be a sequence of distributions with finite support that converges weakly to G. Then worker and firm payoffs in the continuation equilibrium in which other firms offer wages given by the mass points in G_n converge to the payoff functions given by (4.1), (4.2), (4.3) and (4.4) given in Section 3 .*

The complete proof of this theorem is in the on-line appendix.

10. Conclusion

This paper illustrates how a directed search model can be used to model wage competition among firms who can't condition wage payments on worker type. Part of this involves adjusting the directed search model to allow for rich variation in the types of workers and firms. This improves on existing models that use extensive symmetry assumptions that sometimes force the models to behave in counterfactual ways. In the variant proposed here, rich distributions of firm and worker characteristics can be incorporated.

The directed search model does impose some structure on the data. Surprisingly it restricts the relationship between the wage distribution

[18]For example, if the family of payoff functions for a firm determined by the set of worker types is equi-continuous.

and the function relating unemployment duration and exit wage. Some wage distributions (the uniform being an example) have the property that workers who leave unemployment at high wages must also have shorter unemployment duration. This prediction is distinctly different from standard directed search models where unemployment duration and wage must be positively related.

The driving force in the model presented here is the equilibrium of the workers' application sub-game. Contrary to what one might expect, low quality workers do not restrict their applications to low wage firms. On the contrary, low quality workers make applications at all kinds of different wages. The higher the unobservable quality of the worker, the more discriminating the worker is in the wages at which he applies. It is this property that breaks the strong relationship between wage and unemployment probability. Higher quality workers are more likely, everything else constant, to be hired by firms. High quality workers also apply to higher wage firms on average. In this sense high wages and short duration should be related. This relationship is not unambiguous however. As a workers quality rises, he is more likely to be hired at any given firm, but he will also restrict his applications to firms whose wages are higher. This by itself reduces the probability of employment because high wage firms have bigger queues - the usual directed search story.

Finally, the paper suggests how observable data on wages and duration can be used to provide a testable implication of the model.

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